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The KP and more dimensional KdV equations on $A_2^{(1)}$ and $A_3^{(1)}$

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Abstract. The KP equations, which are conjugated to completely integrable flows, on $A_2^{(1)}$ and $A_3^{(1)}$ are examined. These equations fall into different classes determined by the order of the Hamiltonians involved. More dimensional KdV equations appear in this analysis.

1. Introduction

In this paper we describe the KP (Kadomtsev-Petviashvili) equations on the affine Lie algebras $A_2^{(1)}$ and $A_3^{(1)}$ (without central extensions [1]). The equations we study are conjugated to the commutation relations of Hamiltonian flows on certain subalgebras. These subalgebras are equipped with a Poisson structure, for which the flows are completely integrable (even algebraic integrable [2]). The Poisson structure is given by the coadjoint action, derived by means of the Kostant-Adler-Symes theorem [3].

The main argument in favour of this approach is the van Moerbeke-Mumford theorem, which gives the finite zone solutions of the equations as quotients of theta functions on the Jacobian of a Riemann surface [4].

We will focus here on the appearance of the equations in relation with the order of the Hamiltonians from which the equations are derived. It seems indeed that what is normally called the set of KP equations [5, 6] is not a homogeneous set of equations, but falls into well distinguished different classes, determined by the order of the Hamiltonians involved. We stress the fact that our description is different from the one in [5, 6].

It is well known that the KdV and MKdV are describable on $A_1^{(1)}$ [7-10]. Both equations are conjugated to each other. The conjugation defines the Miura transformation [7]. We give a review while fixing the setting for the calculations on $A_2^{(1)}$ and $A_3^{(1)}$.

The MKdV equation is the principal system, while KdV is obtained by conjugation. The general approach goes as follows [11].

Let $\mathscr{L}_{(n)}$ be defined as

$$\mathscr{L}_{(n)}=\left\{\sum_{-\infty}^{m}\xi_i\lambda^i \,\middle|\, \xi_i\in \mathrm{sl}(n,\mathbb{R}),\ m\in\mathbb{Z},\ \lambda\in\mathbb{C}\right\}.$$

The elements of $\mathscr{L}_{(n)}$ are treated as formal series in λ. Hence $\mathscr{L}_{(n)}$ is a $\mathbb{Z}$-graded Lie subalgebra of $A_{n-1}^{(1)}$. We consider the decomposition

$$\mathscr{L}_{(n)}=\mathscr{L}_-\oplus\mathscr{L}_0\oplus\mathscr{L}_+$$

where $\mathscr{L}_-$ is the part consisting of negative powers and $\mathscr{L}_+$ of the positive powers of λ. $\mathscr{L}_0$ is $\mathrm{sl}(n)$. We also consider the subspaces, $m\in\mathbb{Z}$,

$$\mathscr{L}_{(n)}^m=\left\{\sum_{-\infty}^{m}\xi_i\lambda^i \,\middle|\, \xi_i\in\mathrm{sl}(n,\mathbb{R})\right\}.$$

Further, let $\mathrm{sl}(n) = n^- \oplus b$ be the algebra decomposition of $\mathrm{sl}(n)$ $(\mathrm{sl}(n) \equiv \mathrm{sl}(n, \mathbb{R}))$, with respect to a given Cartan subalgebra h of $\mathrm{sl}(n)$; n^- is the nilpotent subalgebra on the negative root spaces and b the Borel subalgebra on the positive root spaces.

The following Lie algebra decomposition of $\mathscr{L}_{(n)}$ will be used throughout this paper:

$$\mathscr{L}_{(n)} = \mathfrak{N} \oplus \mathfrak{K} \qquad \text{with } \mathfrak{N} = \mathscr{L}_- \oplus n^-, \ \mathfrak{K} = b \oplus \mathscr{L}_+.$$

We will further use the standard matrix representation (for which h is diagonal) of $\mathrm{sl}(n)$. As a basis for the invariant polynomials on $\mathrm{sl}(n)$ we will use the homogeneous forms

$$Q_i(\xi) = (1/i)\,\mathrm{Tr}([\xi]^i) \qquad i = 2, \ldots, n;\ \xi \in \mathrm{sl}(n).$$

The quadratic form Q_2 determines the Killing form K.

On $\mathscr{L}_{(n)}$, each of these forms induces an ad $\mathscr{L}_{(n)}$ invariant form by

$$Q_{i,k} = \operatorname*{Res}_{\lambda=0} (\lambda^{k-1} \cdot Q_i).$$

Let $\mathfrak{A}$ denote the algebra (over $\mathbb{R}$) generated by the set $\{Q_{i,k}\}$.

We define the induced bilinear form $\tilde{K}$ on $\mathscr{L}_{(n)}$ in a similar way by

$$\tilde{K} = \operatorname*{Res}_{\lambda=0} \lambda^{-1} K.$$

The form $\tilde{K}$ identifies $\mathfrak{N}^\perp$ with $\mathfrak{K}^*$ (the dual of $\mathfrak{K}$) and $\mathfrak{K}^\perp$ with $\mathfrak{N}^*$.

The subspaces $\mathfrak{K}^{\perp m} = \mathfrak{K}^\perp \cap (\mathscr{L}_0 \oplus \mathscr{L}_+^m)$ are Poisson spaces for the coadjoint action of the simply connected Lie group $\mathfrak{G}(\mathfrak{N})$ with Lie algebra $\mathfrak{N}$ [3, 10]. The functions in $\mathfrak{A}$ are commuting Hamiltonians for this action. The Hamiltonian vector fields are defined by

$$\dot{\xi} = [\mathrm{grad}\, H, \xi] \qquad \xi \in \mathfrak{K}^{\perp m}$$

with $\mathrm{grad}\, H = -P(\mathfrak{K}^m)(\tilde{K}^{-1} \circ d)(H)$, $H \in \mathfrak{A}$ and $P(\mathfrak{K}^m)$ is the projection upon $\mathfrak{K}^m$ [3, 10].

We will more specifically consider the Hamiltonians $\{Q_{i,k}\}$. A subset from this (on a given $\mathfrak{K}^{\perp m}$) are orbit invariants, namely those with zero Hamiltonian vector field.

Let $\pi^k_{k-1} : \mathfrak{K}^{\perp k} \to \mathfrak{K}^{\perp k-1}$ be the projection determined by multiplication with λ^{-1}, followed by a restriction to $\mathfrak{K}^{\perp k-1}$. The coadjoint action of $\mathfrak{G}(\mathfrak{N})$ on $\mathfrak{K}^{\perp k-1}$ is identical with the coadjoint action on the restriction of $\mathfrak{K}^{\perp k}$ to $\lambda \cdot \mathfrak{K}^{\perp k-1}$. This forces us to consider the inverse limit space

$$\mathscr{L}^0 = \varprojlim (\mathfrak{K}^{\perp m}, \pi^m_{m-1}).$$

We endow $\mathscr{L}^0$ with the coordinates $\eta = \Sigma_{i=0}^{\infty} \eta_{-i}\lambda^{-i}$. Any term $\mathfrak{K}^{\perp m}$ is now obtained from $\mathscr{L}^0$ by truncation on the left into $\Sigma_{i=0}^{m} \eta_{-i}\lambda^{-i}$, with η_{-m} in n^+ (the nilpotent algebra constructed on the positive root spaces of $\mathrm{sl}(n)$).

For the sake of simplicity of the discussion we will subsequently consider a $\mathfrak{K}^{\perp m}$ for a given m.

The set of orbit invariants on $\mathfrak{K}^{\perp m}$ is spanned by $\{Q_{i,k} \mid i = 2, \ldots, n;\ k = m + 1, \ldots, i \cdot m + 1\}$. We will only consider the Hamiltonian vector fields with Hamiltonians $Q_{i,m-1}$, $i = 2, \ldots, n$ and $Q_{2,m-2}$.

The Hamiltonian vector field determined by $Q_{2,m-1}$ is given by

$$\dot{\eta} = [\check{\eta}_{-1} + \lambda \cdot \eta_0, \eta] \tag{D}$$

where the symbol $\check{}$ denotes the restriction to b.

We choose the variable x along the integral curves of this vector field and select the Poisson submanifold $\mathfrak{W} \subset \mathfrak{K}^{\perp m}$ by the conditions

$$\eta_0 = \begin{bmatrix} 0 & & & 0 \\ \vdots & & & \\ 0 & 0 & & \\ 1 & 0 & \cdots & 0 \end{bmatrix} \qquad \eta_{-1} = \begin{bmatrix} * & 1 & 0 & \cdots & 0 \\ & & & & \vdots \\ \vdots & & & & 0 \\ & & & & 1 \\ * & & \cdots & & * \end{bmatrix}$$

where η_0 is the base vector of the lowest negative root space and

$$\eta_{-1} \in b^- \oplus \sum_{i=1}^{n-1} e_i$$

with e_i the base vectors for the simple positive root spaces.

The equation (D) will be called the determining equation which will be solved formally (in λ) giving an operator $\sigma : J(x; u_1, \ldots, u_{n-1}) \to \mathscr{L}^0_{(n)}$. $J(x; u_1, \ldots, u_{n-1})$ is the jet bundle of functions in $C^\infty(\mathbb{R}, \mathbb{R}^{n-1})$. The operator σ satisfies the equation

$$D\sigma = [\check{\sigma}_{-1} + \lambda \cdot \sigma_0, \sigma]$$

where D is the total x derivative on the jet bundle. The operator σ depends on constants, the orbit invariants by

$$Q_{i,m}((\sigma)^m) = \sum_{k=-1}^{-\infty} E_{m,k}\lambda^k.$$

$E_{m,k}$ are the orbit invariants [10]. The existence of the solution σ goes back upon a lemma by Wilson [10, 12].

The map into a given $\mathfrak{K}^{\perp m}$ is obtained by simple restriction of σ. This restriction is given by a set of PDE, the invariant equations, which restrict the jet bundle $J(x; u_1, \ldots, u_{n-1})$. Invariance here means invariance with respect to the Hamiltonian vector fields which commute with the x flow [13].

The choice of the jet bundle, together with the map σ, is such that the integral sections (jets of functions in $C^\infty(\mathbb{R}, \mathbb{R}^{n-1})$) are mapped by σ into the integral curves of the vector field (D). We call such an operator σ an holonomic momentum operator. The commuting Hamiltonian vector fields, on $\mathfrak{K}^{\perp m}$, determine, by derivation of σ, evolution equations on the jet bundle [10, 11].

For $n = 2$ we obtain the MKdV equation from the quadratic Hamiltonian $Q_{2,m-2}$. Let

$$\sigma = \sum_{i=0}^{\infty} \sigma_{-i}\lambda^{-i} \qquad \text{with} \qquad \sigma_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \check{\sigma}_{-1} = \begin{pmatrix} v & 1 \\ 0 & -v \end{pmatrix}$$

v is a function of x in $C^\infty(\mathbb{R}, \mathbb{R})$, and hence the target variable of the jet bundle $J(x; v)$.

Solving the determining equation (D) with the conditions

$$\tfrac{1}{2}\operatorname{Tr}(\sigma \cdot \sigma) = \sum_{k=-1}^{-\infty} E_{2,k}\lambda^k$$

with $E_{2,-1} = 1$, $E_{2,-2} = 0$ yields

$$\sigma_{-1} = \begin{bmatrix} v & 1 \\ \tfrac{1}{2}(-v^2 + v_x) & -v \end{bmatrix} \qquad \sigma_{-2} = \begin{bmatrix} \tfrac{1}{2}(v_{xx} - 2v^3) & -\tfrac{1}{2}(v^2 + v_x) \\ * & -\tfrac{1}{2}(v_{xx} - 2v^3) \end{bmatrix}.$$

Here v_x means partial derivative with respect to x. The MKdV is given by the Hamiltonian vector field

$$D_t\sigma = [\check{\sigma}_{-2} + \lambda\sigma_{-1} + \lambda^2\sigma_0, \sigma]$$

giving

$$v_t = \tfrac{1}{4}(v_{xxx} - 6v^2 \cdot v_x).$$

From the commutativity of the flows one derives the principal Bäcklund equations (or zero curvature equations)

$$D\check{\sigma}_{-i} - D_{t_{i-1}}\check{\sigma}_{-1} + [\check{\sigma}_{-i}, \check{\sigma}_{-1}] = 0 \qquad (\mathrm{B_p})$$

for each i and $t_0 \equiv x$.

The $\check{\sigma}_{-i}$ can be expressed by means of the operator

$$R = \begin{bmatrix} -\frac{1}{2}(2v - D) & 1 \\ 0 & \frac{1}{2}(2v - D) \end{bmatrix}$$

and potentials ξ_i, such that

$$\check{\sigma}_{-i} = R(\xi_i).$$

The conjugated system is obtained from the transformation on the momentum operator

$$\sigma \mapsto \varphi \cdot \sigma \cdot \varphi^{-1}$$

with

$$\varphi = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

The conjugated Bäcklund equations are

$$D_{t_i}\Psi_1 - D\Psi_{i+1} + [\Psi_1, \Psi_{i+1}] = 0 \qquad (\mathrm{B_{conj}})$$

with

$$\Psi_i = \varphi \cdot \check{\sigma}_{-i}\varphi^{-1} + D_{t_{i-1}}\varphi \cdot \varphi^{-1}.$$

From

$$\Psi_1 = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}$$

one gets the Miura transformation [7]

$$u = v^2 - v_x.$$

The Ψ_i are also written as

$$\Psi_i = P(\tilde{\xi}_i)$$

with

$$P = \begin{bmatrix} -\frac{1}{2}D & 1 \\ -\frac{1}{2}D^2 + u & \frac{1}{2}D \end{bmatrix}$$

and

$$\tilde{\xi}_i(u = v^2 - v_x) = \xi_i.$$

The potential $-\frac{1}{2}u$, for Ψ_2, determines the KdV equation [9]:

$$D_t\Psi_1 - D\Psi_2 + [\Psi_1, \Psi_2] = 0$$

or

$$u_t = \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}u \cdot u_x.$$

The higher-order equations are then given by the higher $Q_{2,m-k}$, $k > 2$.

The operator takes values in $\mathscr{L}^0$ and is restricted to $\mathfrak{K}^{\pm m}$ by the condition $\sigma_{-m}|_h = 0$, h being the Cartan subalgebra. This condition becomes $(D-2v)\xi_m = 0$, with ξ_m the mth potential determining σ_{-m}. We remark that this PDE depends on the orbit invariants $E_{2,k}$.

The restriction of σ to a $\mathfrak{K}^{-m}$ defines a regular difference operator in the sense of van Moerbeke and Mumford [4]. Hence the solutions are rational functions on a complex torus, the Jacobian of the Riemann surface $\det|\sigma - z \cdot Id| = 0$. This is an hyperelliptic curve. The solutions are the finite zone solutions [14, 15].

We will follow the same scheme to derive the equations on $A_2^{(1)}$ and $A_3^{(1)}$.

2. The equations on $A_2^{(1)}$

We define the principal system by the operator

$$\sigma : J(x; u, v) \to \mathscr{L}^0_{(3)}$$

together with the conditions

$$\sigma_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \check{\sigma}_{-1} = \begin{pmatrix} u & 1 & 0 \\ 0 & -u-v & 1 \\ 0 & 0 & v \end{pmatrix}.$$

The determining equation is solved with the conditions:

$$\mathrm{Tr}(\sigma \cdot \sigma) = \sum_{i=1}^{\infty} E_{2,-i}\lambda^{-i} \qquad \mathrm{Tr}(\sigma \cdot \sigma \cdot \sigma) = \sum_{i=1}^{\infty} E_{3,-i}\lambda^{-i}$$

with

$$E_{2,-1} = E_{3,-1} = E_{2,-2} = 0 \qquad E_{3,-2} = 3 \qquad E_{3,-3} = 0.$$

The commuting flows produce the Bäcklund equations

$$D_t\check{\sigma}_{-1} - D\phi + [\check{\sigma}_{-1}, \Phi] = 0 \tag{B$_\mathrm{p}$}$$

where Φ takes values in b.

It follows from these equations that Φ is necessarily determined by means of two potentials (α, β) and the operator R:

$$\Phi = R\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with

$$R_{11} = (\tfrac{2}{3}D^2 - 2uD + \tfrac{2}{3}v_x - \tfrac{2}{3}v^2 - \tfrac{2}{3}u_x + \tfrac{4}{3}u^2,\ u - D)$$
$$R_{12} = (-D - v - u,\ 1)$$
$$R_{13} = (1, 0)$$
$$R_{22} = (-\tfrac{1}{3}D^2 + uD - \tfrac{1}{3}v_x + \tfrac{1}{3}v^2 + \tfrac{1}{3}vu + \tfrac{1}{3}u_x - \tfrac{2}{3}u^2,\ -u - v)$$
$$R_{23} = (0, 1)$$
$$R_{33} = (-\tfrac{1}{3}D^2 + uD - \tfrac{1}{3}v_x + \tfrac{1}{3}v^2 + \tfrac{1}{3}vu + \tfrac{1}{3}u_x - \tfrac{2}{3}u^2,\ D + v)$$
$$R_{21} = R_{31} = R_{32} = (0, 0).$$

On $\mathfrak{K}^{\perp m}$ we consider the following Hamiltonian flows:

Flow	Hamiltonian	Potentials
x	$Q_{2,m-1}$ (quadratic)	$(0, 1)$
y	$Q_{3,m-1}$ (cubic)	$(1, u)$
t	$Q_{2,m-2}$ (quadratic)	(α_1, β_1)

with

$$\alpha_1 = \tfrac{1}{3}(v_x - v^2 - uv - u^2 - u_x)$$

$$\beta_1 = -u\alpha_1 - \tfrac{1}{3}(uv_x - v^2u - vu^2 + u_{xx} + 2uu_x).$$

The conjugated system is given by $\sigma_{\mathrm{conj}} = \varphi \cdot \sigma \cdot \varphi^{-1}$ with

$$\varphi = \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ u^2 + u_x - v & -v & 1 \end{bmatrix}.$$

From

$$\Psi_1 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ F & G & 0 \end{pmatrix} = \varphi \cdot \check{\sigma}_{-1} \cdot \varphi^{-1} + D\varphi \cdot \varphi^{-1}$$

one finds the Miura transformation

$$F = u_{xx} + 2uu_x + uv_x - u^2v - uv^2 \qquad G = u_x - v_x + u^2 + uv + v^2. \quad (\mathfrak{M}\text{-}1)$$

The conjugated Bäcklund equations

$$D_t\Psi_1 - D\Psi + [\Psi_1, \Psi] = 0 \qquad (\mathrm{B_{conj}})$$

determine the Ψ by means of the transformed potentials $(\tilde{\alpha}, \tilde{\beta})$ and the operator P:

$$\Psi = P\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix}$$

with

$$P_{11} = (\tfrac{2}{3}(D^2 - G), -D)$$

$$P_{12} = (-D, 1)$$

$$P_{13} = (1, 0)$$

$$P_{21} = (\tfrac{1}{3}(2D^3 - 2G \cdot D + 3F - 2G_x), -D^2)$$

$$P_{22} = (\tfrac{1}{3}(-D^2 + G), 0)$$

$$P_{23} = (0, 1)$$

$$P_{31} = (\tfrac{1}{3}(2D^4 - 2G \cdot D^2 + 3F \cdot D - 4G_xD + 3F_x - 2G_{xx}), -D^3 + F)$$

$$P_{32} = (\tfrac{1}{3}(D^3 - G \cdot D + 3F - G_x), -D^2 + G)$$

$$P_{33} = (\tfrac{1}{3}(-D^2 + G), D).$$

The potentials $(\tilde{\alpha}, \tilde{\beta})$ relate to the (α, β) by

$$\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} = \begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{T}$$

The considered flows in this conjugated system are

Potentials	Transformed potentials	Flow
$(0, 1)$	$(0, 1)$	x
$(1, u)$	$(1, 0)$	y
(α_1, β_1)	$(-\frac{1}{3}G, -\frac{1}{3}F)$	t

With

$$\Psi_2 = P\begin{pmatrix}1\\ 0\end{pmatrix} \qquad \Psi_3 = P\begin{pmatrix}-\frac{1}{3}G\\ -\frac{1}{3}F\end{pmatrix}$$

the Bäcklund equations are

$$D_y\Psi_1 - D\Psi_2 + [\Psi_1, \Psi_2] = 0 \tag{B-1}$$

$$D_t\Psi_1 - D\Psi_3 + [\Psi_1, \Psi_3] = 0 \tag{B-2}$$

$$D_y\Psi_3 - D_t\Psi_2 + [\Psi_3, \Psi_2] = 0. \tag{B-3}$$

Equation (B-1) gives

$$DF = \tfrac{1}{2}(D_yG + D^2G). \tag{B-1-1}$$

The contact transformation $J(x, y; w) \to J(x; F, G)$ given by

$$G = w_x \qquad F = \tfrac{1}{2}(w_y + w_{xx}) \tag{B-1-2}$$

solves (B-1-1).

Equation (B-1) then reduces to the Boussinesq equation

$$w_{yy} = \tfrac{1}{3}(-w_{xxxx} + 4w_x w_{xx}) \tag{B-1-3}$$

while (B-2) becomes

$$w_t = \tfrac{1}{3}(w_{xxy} - 2w_x \cdot w_y). \tag{B-1-4}$$

This last equation is determined by (B-2) up to a constant $(w_t + C)$, which we have set equal to zero, fixing the initial condition of w_t.

Given the equations (B-1-3) and (B-1-4) one finds that equation (B-3) is identically satisfied.

The Miura transformation now is

$$\begin{aligned} w_y &= u_{xx} + v_{xx} + 2uu_x - 2vv_x + uv_x - vu_x \\ w_x &= u_x - v_x + u^2 + uv + v^2 \end{aligned} \tag{$\mathfrak{M}$-2}$$

and relates solutions of the (completely integrable) principal system to solutions of (B-1-3) and (B-1-4).

Equation (B-1-4) is an evolution equation lying on the subspace of $J(x, y; w)$ determined by the Boussinesq equation (B-1-3), together with its prolongations. Because (B-1-4) is determined by the quadratic Hamiltonian $Q_{2,m-2}$, it generalises the KdV equation into two (spacelike) dimensions.

The term Ψ_1 determines the operator

$$D-\Psi_1=\begin{bmatrix} D & -1 & 0 \\ 0 & D & -1 \\ -\frac{1}{2}(w_y+w_{xx}) & -w_x & D \end{bmatrix}$$

which is the linearisation of

$$L=-\tfrac{1}{2}(w_y+w_{xx})-w_xD+D^3.$$

This operator gives the Lax formulation of the equations (B-1-3) and (B-1-4).

3. The equations on $A_3^{(1)}$

The principal system is determined by

$$\sigma\colon J(x;\,u,\,v,\,w)\to\mathscr{L}^0_{(4)}$$

with the conditions

$$\sigma_0=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \check{\sigma}_{-1}=\begin{bmatrix} u & 1 & 0 & 0 \\ 0 & -u+v & 1 & 0 \\ 0 & 0 & -v+w & 1 \\ 0 & 0 & 0 & -w \end{bmatrix}.$$

The determining equation (D) is solved by means of

$$\mathrm{Tr}([\sigma]^m)=\sum_{i=2}^{\infty}E_{m,-i}\lambda^{-i} \qquad m=2,3,4$$

with $E_{m,-2}=0$, $m=2,3,4$; $E_{3,-3}=E_{4,-4}=0$, $E_{4,-3}=4$.

The commuting flows give

$$D_t\check{\sigma}_{-1}-D\phi+[\check{\sigma}_{-1},\,\phi]=0 \qquad (\mathrm{B_p})$$

where ϕ takes values in b. From this we derive the operator form

$$\phi_i=R\begin{pmatrix}\alpha_i\\ \beta_i\\ \gamma_i\end{pmatrix}$$

where R is a differential operator with values in b which is too long to write down here. The $(\alpha_i,\beta_i,\gamma_i)$ are the potentials.

For a fixed m, the flows on $\mathfrak{K}^{\perp m}$ in which we are interested are:

Flow	Hamiltonian	Potentials
x	$Q_{2,m-1}$ (quadratic)	$(0, 0, 1)$
y	$Q_{3,m-1}$ (cubic)	$(0, 1, -v)$
z	$Q_{4,m-1}$ (quartic)	$(1, -u, \gamma_2)$
t	$Q_{2,m-2}$ (quadratic)	$(\alpha_3, \beta_3, \gamma_3)$

with

$$\gamma_2 = w^2 - wv + v^2 + v_x + w_x + 3\alpha_3$$

$$\alpha_3 = \tfrac{1}{4}(-u^2 + uv - v^2 + vw - w^2 - w_x - v_x - u_x)$$

$$\beta_3 = \tfrac{1}{8}(2u^3 - 4u^2 v + 4uv^2 - 2uvw + 2uw^2 + 2uw_x + 5uv_x - 4uu_x - 2v^2 w + 2vw^2 + vw_x - 2vv_x + vu_x + 2ww_x + wv_x + w_{xx} - v_{xx} - 3u_{xx})$$

$$\begin{aligned}\gamma_3 = \tfrac{1}{32}(&5u^4 - 10u^3 v + 15u^2 v^2 - 10u^2 vw + 10u^2 w^2 + 10u^2 w_x - 10u^2 v_x + 10u^2 u_x - 10uv^3 \\ &+ 10uv^2 w - 10uvw^2 - 10uvw_x + 10uvv_x - 10uvu_x + 10uv_{xx} - 20uu_{xx} - 3v^4 \\ &+ 14v^3 w - 17v^2 w^2 - 14v^2 w_x + 2v^2 v_x + 10v^2 u_x + 6vw^3 + 6vww_x + 2vwv_x \\ &- 10vwu_x - 6vw_{xx} + 8vv_{xx} + 10vu_{xx} - 3w^4 - 6w^2 w_x - 2w^2 v_x \\ &+ 10w^2 u_x + 12ww_{xx} - 6wv_{xx} + 9w_x w_x - 14w_x v_x + 10w_x u_x + v_x v_x + 10v_x u_x \\ &- 15u_x u_x + 6w_{xxx} + 2v_{xxx} - 10u_{xxx}).\end{aligned}$$

The conjugated system is determined by $\varphi \cdot \sigma \cdot \varphi^{-1}$ with

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u & 1 & 0 & 0 \\ u^2 + u_x & v & 1 & 0 \\ u^3 + 3uu_x + u_{xx} & u^2 - uv + v^2 + u_x + v_x & w & 1 \end{bmatrix}.$$

This determines with

$$\Psi_1 \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ F_1 & F_2 & F_3 & 0 \end{bmatrix} = \varphi \cdot \check{\sigma}_{-1} \varphi^{-1} + D\varphi \cdot \varphi^{-1}$$

the Miura transformation

$$\begin{aligned} F_1 &= u^2 vw - u^2 w^2 - u^2 w_x + u^2 v_x - uv^2 w + uvw^2 + uvw_x - 2uvv_x + 2uvu_x \\ &\quad - uv_{xx} + 2uu_{xx} - v^2 u_x + vwu_x - w^2 u_x - w_x u_x + u_{xxx} - v_x u_x + 2u_x u_x \\ F_2 &= u^2 v - uv^2 - 2uv_x + 4uu_x + v^2 w - vw^2 - vw_x + 2vv_x - vu_x + v_{xx} + 2u_{xx} \\ F_3 &= u^2 - uv + v^2 - vw + w^2 + w_x + v_x + u_x. \end{aligned} \qquad (\mathfrak{M}\text{-}1)$$

The conjugated Bäcklund equations are

$$D_{t_{i-1}} \Psi_1 - D\Psi_i + [\Psi_1, \Psi_i] = 0 \qquad (\mathrm{B}_{\mathrm{conj}})$$

with $t_0 \equiv x$ and again with

$$\Psi_{i+1} = P \begin{bmatrix} \tilde{\alpha}_i \\ \tilde{\beta}_i \\ \tilde{\gamma}_i \end{bmatrix}$$

where P is a differential operator with values in sl(4), which we will not write down.

The potentials $(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i)$ are related to the $(\alpha_i, \beta_i, \gamma_i)$ by

$$\begin{bmatrix} \tilde{\alpha}_i \\ \tilde{\beta}_i \\ \tilde{\gamma}_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ u^2 + u_x & v & 1 \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{bmatrix}.$$

The considered flows are

Flow	Potentials	Transformed potentials
x	$(0,0,1)$	$(0,0,1)$
y	$(0,1,-v)$	$(0,1,0)$
z	$(1,-u,\gamma_2)$	$(1,0,\frac{1}{4}F_3)$
t	$(\alpha_3,\beta_3,\gamma_3)$	$(-\frac{1}{4}F_3, -\frac{1}{4}F_2+\frac{1}{8}F_{3,x}, \frac{1}{32}(8F_1+4F_{2,x}-6F_{3,xx}+3(F_3)^2))$

The Bäcklund equations are the following six equations:

$$B_{ij} \equiv D_{t_i}\Psi_j - D_{t_j}\Psi_i + [\Psi_j, \Psi_i] = 0 \qquad \qquad (\mathrm{B}_{\mathrm{conj}})$$
$$i,j = 1,2,3,4 \ (i \neq j) \qquad t_1 = x \qquad t_2 = y \qquad t_3 = z \qquad t_4 = t.$$

Equation (B_{12}) gives

$$DF_2 = D^2F_3 + \tfrac{1}{2}D_yF_3. \qquad (\mathrm{B}_{12}\text{-}1)$$

This equation is solved by means of the contact transformation $J(x,y;F_1,G) \to J(x;F_1,F_2,F_3)$ given by

$$F_3 = DG \qquad F_2 = D^2G + \tfrac{1}{2}D_yG. \qquad (\mathrm{B}_{12}\text{-}2)$$

On this jet bundle equation (B_{12}) becomes

$$DF_1 = \tfrac{1}{4}(D_{yy}G + 2D^4G + D^2D_yG - 2DG\cdot D^2G) \qquad (\mathrm{B}_{12}\text{-}3)$$

$$D_yF_1 = \tfrac{1}{4}(DD_{yy}G + D^3D_yG + D_yG\cdot D^2G) \qquad (\mathrm{B}_{12}\text{-}4)$$

which yields by integrability

$$D_y(D_{yy}G + D^4G - 2DG\cdot D^2G) = D(D_yG\cdot D^2G). \qquad (\mathrm{B}_{12}\text{-}5)$$

Equation (B_{13}) gives the relation

$$DF_1 = \tfrac{1}{12}(5D^4G + 3D^3D_yG - 3DG\cdot D^2G + 4DD_zG) \qquad (\mathrm{B}_{13}\text{-}1)$$

which we solve by means of the contact transformation $J(x,y,z;G) \to J(x,y;F_1,G)$ with

$$F_1 = \tfrac{1}{12}[5D^3G + 3DD_yG - \tfrac{3}{2}(DG)^2 + 4D_zG]. \qquad (\mathrm{B}_{13}\text{-}2)$$

Equation (B_{13}) reduces to

$$D_{yy}G = \tfrac{1}{3}(4DD_zG - D^4G + 3DG\cdot D^2G) \qquad (\mathrm{B}_{13}\text{-}3)$$

$$D_yD_zG = -\tfrac{1}{4}(2D^3D_yG - 3DD_yG\cdot DG - 3D_yG\cdot D^2G) \qquad (\mathrm{B}_{13}\text{-}4)$$

$$D_{zz}G = -\tfrac{1}{16}(-2D^6G - 4D^3D_zG - 9DD_yG\cdot D_yG - 9D^4G\cdot DG$$
$$+ 18D^3G\cdot D^2G - 9D^2G(DG)^2). \qquad (\mathrm{B}_{13}\text{-}5)$$

Equation $(\mathrm{B}_{13}\text{-}3)$ is the usual KP equation [14, 16]. Then, given the former equations, $(\mathrm{B}_{13}\text{-}4)$ reduces to

$$D_tG = -\tfrac{1}{48}(2D^5G - 2D^3D_zG + 20DG\cdot D_zG$$
$$+ \tfrac{15}{2}D_yG\cdot D_yG - 5D^3G\cdot DG - \tfrac{15}{2}D^2G\cdot D^2G). \qquad (\mathrm{B}_{14}\text{-}1)$$

This last equation is (similar to the $\mathscr{L}_{(3)}$ case) defined up to a constant which we have set equal to zero.

The remaining Bäcklund equations are identically satisfied by the set $((\mathrm{B}_{12}\text{-}5)$, $(\mathrm{B}_{13}\text{-}3)$, $(\mathrm{B}_{13}\text{-}4)$, $(\mathrm{B}_{13}\text{-}5)$, $(\mathrm{B}_{14}\text{-}1))$ and their prolongations. The equations $((\mathrm{B}_{12}\text{-}5)$, $(\mathrm{B}_{13}\text{-}3)$, $(\mathrm{B}_{13}\text{-}4)$, $(\mathrm{B}_{13}\text{-}5))$ are submanifolds of $J(x,y,z;G)$, while $(\mathrm{B}_{14}\text{-}1)$ is a symmetry equation. It is again natural to consider this equation as a generalisation of the KdV equation into three spacelike dimensions.

Equation (B_{12}-5) generalises the Boussinesq equation, while (B_{13}-3) is a new equation obtained from the commutation of the z flow with the two former flows (the x and y flows). This equation, which is the genuine Kadomtsev–Petviashvili equation, belongs to the larger set given by (B_{13}-4) and (B_{13}-5).

The operator which now defines the Lax representation is

$$L=-\tfrac{1}{12}[5G_{xxx}+3G_{yx}-\tfrac{3}{2}(G_x)^2+4G_z]-(G_{xx}+\tfrac{1}{2}G_y)D-G_xD^2+D^4.$$

4. Conclusions and remarks

(a) For any $\mathscr{L}_{(n)}$ the conjugated Bäcklund equations are normally called KP equations [5, 6]. The main purpose of this paper is to show, for $n=3, 4$, that these equations naturally fall into several classes. The equations are related to the commutation of Hamiltonian flows determined by Hamiltonians of a given order, using the Kostant–Adler–Symes theorem. The technique of enlarging the number of the space variables is used to reduce the number of independent functions involved. By doing so, and by following a given order of the flows (on a $\mathfrak{K}^{\perp m}$), one finds that the Boussinesq equation appears from a cubic Hamiltonian, while the genuine KP equation appears from a quartic Hamiltonian.

The first next quadratic Hamiltonian, after the x flow, determines for $n=2$ the KdV equation, while for $n=3$ or 4 it determines an evolution equation defined on a space of $(n-1)$ variables.

(b) The invariant submanifolds (PDE which restrict the holonomic momentum operator σ to a given $\mathfrak{K}^{\perp m}$) are given by $\sigma_{-m}|_h=0$ where h is the (diagonal) Cartan subalgebra of sl(n). The restricted σ corresponds to a regular difference operator [4]. The algebraic curve is an n-fold covering of the complex plane.

Because the algebraic curve $\det|\sigma - z\circ id|=0$ is invariant under conjugation one may define this curve directly from $\varphi\cdot\sigma\cdot\varphi^{-1}$.

The operator σ determines a local diffeomorphism of the PDE and an orbit in $\mathfrak{K}^{\perp m}$ fixed by the orbit invariants. The symplectic structure of the coadjoint structure pulls back upon the PDE (if $n=3$ the Boussinesq equation, if $n=4$ the equations (B_{12}-5), (B_{13}-3)–(B_{13}-5) restricted by the invariant PDE [10]).

Each of the equations has an infinite set of conservation laws [10] which we have not analysed.

(c) We have preferred to present the calculation rather than the theorems. One could, for all the steps we made, formulate appropriate theorems, which are easily generalised to sl(n) for any n. Because the calculations are long we are not able to give all the details, but the omitted steps should easily be reconstructed from the data we gave. The calculations were carried out on an IBM 4381 using REDUCE.

We finally remark that the same calculations can be carried out for any real simple Lie algebra using the theorems of [3].

References

[1] Macdonald I G 1981 *Lecture Notes in Mathematics* vol 901 (Berlin: Springer)
[2] van Moerbeke P 1983 *Proc. Int. Congr. on Mathematics, Warsaw* p 881
[3] Adler M and van Moerbeke P 1980 *Adv. Math.* **38** 267
[4] van Moerbeke P and Mumford D 1979 *Acta Math.* **143** 93
[5] Segal G and Wilson G *IHES Publ. Math.* **61** 5

[6] Shiota T 1986 *Inv. Math.* **83** 333
[7] Drinfel'd V G and Sokolov V V 1981 *Dokl. Akad. Nauk SSSR* **258** 11
[8] Flaschka H, Newell A C and Ratiu T 1983 *Physica* **9D** 300, 324
[9] Dhooghe P F 1985 *Rep. Math. Phys.* **21** 159
[10] Dhooghe P F 1985 *Lecture Notes in Physics* vol 239 (Berlin: Springer) p 232
[11] Dhooghe P F 1986 *Acta Appl. Math.* **5** 181
[12] Wilson G 1981 *Ergodic Theor. Dyn. Syst.* **1** 361
[13] Gel'fand I M and Dikii L A 1978 *Funct. Anal. Appl.* **12** no 2, 8
[14] Dubrovin B A 1981 *Usp. Mater. Nauk* **36** no 2, 11
[15] Dubrovin B A 1977 *Funk. Anal. Pril.* **11** no 4, 28
[16] Mulase M 1983 *Math. Sci. Res. Inst.* 053-83